Uniformity of mixing transformations with infinite measure

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(Received Oct. 5, 1981)

Introduction. Mixing transformations on a completely regular topological measure space with infinite measure are not necessarly ergodic [7]. The purpose of this note is to treat that some attached conditions of mixing transformations in order to imply the ergodicity.

Let $(X, \mathfrak{B}, \mu)$ be a topological measure space of a completely regular topological space X, the σ-field $\mathfrak{B}$ of Borel sets of X, and a measure μ on $\mathfrak{B}$, where the measure μ assumed to be non-negative, σ-additive, locally finite and tight in the sense that $\mu(A)=\sup\{\mu(K); K\subset A$, K is compact$\}$ for every $A\in\mathfrak{B}$. A set $A\in\mathfrak{B}$ will be almost clopen if it characteristic function is μ-almost everywhere (in what follows it denotes abbreviated a.e.) continuous. Let $\mathfrak{K}$ be the set of all clopen sets of $\mathfrak{B}$. It is clear that $\mathfrak{K}$ is a field but not σ-field. It is known that $\mathfrak{K}$ is μ-metrically dense in $\mathfrak{B}$ [7].

An endomorphism T of X is called *mixing* if it has the following properties [6, 7].

(i) T is continuous (a.e.), and measure preserving.

(ii) There is a sequence of $\mathfrak{K}$-subsets $H_k\in\mathfrak{K}$ $(k=1, 2, 3, \ldots)$ of X with finite measure such that $X=\bigcup_{k=1}^{\infty} H_k$ (a.e.).

(iii) There is a sequence of positive numbers ρ_n $(n=1, 2, 3, \ldots)$ such that the mixing relation $\lim_{n\to\infty} \rho_n\ \mu(A\cap T^{-n}B)=\mu(A)\ \mu(B)$ holds for any $\mathfrak{K}$-sets A and B included in some H_k.

T-invariant subsets. Let T be a measure preserving transfmation on X. A T-invariant $\mathfrak{B}$-set A which has a $\mathfrak{B}$-subset D such that $\bigcup_{-\infty}^{\infty} T^n D=A$ (a.e.) and $0<\mu(D)<\varepsilon$ for any small positive number ε. Such a set A is called *T-osmotic* and the set D is called *generetor* of A. A T-invariant $\mathfrak{B}$-set is called *T-conservative* if it has no wandering set. A T-invariant $\mathfrak{B}$-set is called *T-ergodic* if it has no non-trivial T-invariant subset. A $\mathfrak{B}$-set D is called *T-transitive* if for any μ-non-null subsets E and F of D there is an integer n such that $\mu(E\cap T^{-n}F)\neq 0$. Then it is clear that the followings.

LEMMA 1. *T-ergodic set is also T-osmotic.*

LEMMA 2. *T-osmotic set which has T-transitive generator is T-ergodic.*

PROOF. Let A be a T-osmotic set having T-transitive generator D. If A is not T-ergodic, i.e., A has non-trivial T-invariant $\mathfrak{B}$-subset B. Let $C=A\setminus B$ ($A\setminus B$ denotes the set of elements which belong to A but not B), then $\mu(C)\neq 0$. Let $D_1=D\cap B$ and $D_2=D\cap C$, then clearly $\mu(D_1)\neq 0$, $\mu(D_2)\neq 0$, and D_1, D_2 are generators of B, C respectively. Since D is T-transitive, hence there is a positive integer n such that $\mu(D_1\cap T^{-n}D_2)\neq 0$, i.e., $\mu(B\cap C)\neq 0$. It is a contradiction.

LEMMA 3. *T-osmotic set is also T-conservative.*

PROOF. Let A be not T-conservative, and its wandering set denotes $\{T^nE\}$; $n=0,\pm 1,\pm 2,\ldots$ (where $T^0E=E$). Now we put any $\mathfrak{B}$-subset F of A such that $0<\mu(F)<\mu(E)$. Let $F\cap T^nE=F_n$, then clearly F_n's are mutually disjoint and $T^{-n}F_n\subset E$. Since $\mu\left(\bigcup_{-\infty}^{\infty}T^{-n}F_n\right)\leqq\sum_{-\infty}^{\infty}\mu(F_n)=\mu(F)<\mu(E)$, hence we can put a $\mathfrak{B}$-set G such that $G\subset E\setminus\bigcup_{-\infty}^{\infty}T^{-n}F_n$. Then we have

$$T^mG\subset T^mE\setminus T^m\left(\bigcup_{-\infty}^{\infty}T^{-n}F_n\right)\subset T^mE\setminus T^m(T^{-m}F_m)=T^mE\setminus F_m=T^mE\setminus F \text{ for any integer } m,$$

i.e., $(T^mG)\cap F=\phi$ or equivalently $G\cap(T^mF)=\phi$ (ϕ denotes empty set). It implies that $\bigcup_{-\infty}^{\infty}T^nF$ is non-trivial T-invariant subset of A. This shows that A is not T-osmotic.

Mixing transformations. The sequence $\{\rho_n\}$ of positive numbers in the definition of mixing property(iii) is considered as some sort of dilution factor and asymptotically independent of the choice of the $H_k\in\mathfrak{K}$ in the definition of mixing property(ii) [6]. It is know that the behaviour of the dilution sequence $\{\rho_n\}$ of mixing transformation concerned to ergodicity of the transformation [6, 7].

Now we show the followings with regared to the dilution sequence of mixing transformations.

THEOREM 1. *Let T be an invertible mixing transformation governed by the sequence $\{\rho_n\}$. If there exist T-osmotic set, then* $\sum_{n=1}^{\infty}\rho_n^{-1}=\infty$.

PROOF. Let A be a T-osmotic set, then there is a $H\in\mathfrak{K}$ in the definition of mixing property(ii) such that $\mu(H\cap A)\neq 0$. From mixing relation(iii) there is a sufficiently large integer N such that $\rho_n^{-1}>\mu(H\cap T^{-n}H)/2\mu(H)^2$ for any integer $n\geqq N$. Now, if we suppose $\sum_{1}^{\infty}\rho_n^{-1}<\infty$, then there is an integer $m\geqq N$ such that such that $\sum_{m}^{\infty}\mu(H\cap T^{-n}H)<\varepsilon$ for any positive number ε. Let $\varepsilon<\frac{1}{2}\mu(H\cap A)$ and $K=(H\cap A)\setminus\bigcup_{m}^{\infty}(H\cap T^{-n}H)$, then $\mu(K)>0$ and $K\cap T^{-n}K=\phi$ for any integer $n\geqq m$. Consequently, $K\cap T^{\pm n}K=\phi$ for $n\geqq m$. Let G be any $\mathfrak{B}$-subset of A such that $0<\mu(G)<\frac{1}{2m}\mu(K)$, If $G\subset K$, then $\mu\left(\left(\bigcup_{-\infty}^{\infty}T^nG\right)\cap K\right)<\mu(K)$, i.e., $\mu\left(A\setminus\bigcup_{-\infty}^{\infty}T^nG\right)\neq 0$. It contradicts to T-osmoticity of A. Hence G is not contained in K. Let $D_1=G\cap K$ and $G_1=G\cap\left(A\cap\left(\bigcup_{m}^{\infty}H\cap T^{-n}H\right)\right)$, then $\mu(G_1)\neq 0$. Since $\mu\left(K\setminus\bigcup_{-\infty}^{\infty}T^nD_1\right)>\mu(K)-2m\mu(D_1)$

$>\mu(K)-2m\mu(G)>0$, hence $K\setminus\bigcup_{-\infty}^{\infty}T^nD_1\neq\phi$(a.e.). Let $G_2=G_1\setminus\bigcup_{-\infty}^{\infty}T^nD_1$. If $\mu(G_2)=0$, then $G\subset\bigcup_{-\infty}^{\infty}T^nD_1$ (a.e.), i.e., $\bigcup_{-\infty}^{\infty}T^nG=\bigcup_{-\infty}^{\infty}T^nD_1$. Since $A\setminus\bigcup_{-\infty}^{\infty}T^nD_1\supset K\setminus\bigcup_{-\infty}^{\infty}T^nD_1\neq\phi$(a.e.), hence $A\setminus\bigcup_{-\infty}^{\infty}T^nG$(a.e.). It contradicts to T-osmoticity of A. Thus $\mu(G_2)\neq 0$. Let $D_2=K\cap T^{-1}G_2$, then

$\mu\left(K\setminus\bigcup_{i=1}^{2}\left(\bigcup_{-\infty}^{\infty}T^nD_j\right)\right)>\mu(K)-2m\{\mu(D_1)+\mu(D_2)\}>\mu(K)-2m\mu(G)>0$. Let $G_3=G_2\setminus\bigcup_{-\infty}^{\infty}T^mD_2$. Then $TD_2=TK\cap G_2\subset G_2\setminus G_3$, thus $\mu(D_2)+\mu(D_3)\leqq\mu(G_2)$. If $\mu(G_3)=0$, then $G\subset\left(\bigcup_{-\infty}^{\infty}T^nD_1\right)\cup\left(\bigcup_{-\infty}^{\infty}T^nD_2\right)$. Hence

$$A\setminus\bigcup_{-\infty}^{\infty}T^nG\supset A\setminus\bigcup_{j=1}^{2}\left(\bigcup_{-\infty}^{\infty}T^nD_j\right)\supset K\setminus\bigcup_{j=1}^{2}\left(\bigcup_{-\infty}^{\infty}T^nD_j\right)\neq\phi\ \text{(a.e.)}.$$

It contradicts to T-osmoticity of A. Thus $\mu(G_3)\neq 0$. Let $D_3=K\cap TG_3$, then

$\mu\left(K\setminus\bigcup_{j=1}^{3}\left(\bigcup_{-\infty}^{\infty}T^nD_j\right)\right)>\mu(K)-2m\left\{\sum_{j=1}^{3}\mu(D_j)\right\}>\mu(K)-2m\{\mu(D_1)+\mu(D_2)+\mu(G_3)\}>\mu(K)-2m\mu(D_1)+\mu(G_2)\}>\mu(K)-2m\mu(G)>0$. Inductvely, let $D_{2l}=K\cap T^{-1}G_{2l}$, $G_{2l+1}=G_{2l}\setminus\bigcup_{-\infty}^{\infty}T^nD_{2l}$, $D_{2l+1}=K\cap TG_{2l+1}$, $G_{2l+2}=G_{2l+1}\setminus\bigcup_{-\infty}^{\infty}T^nD_{2l+1}$ $(l=1,2,3,\ldots)$, then $\mu(G_{2l+1})\neq 0$, $\mu(G_{2l+2})\neq 0$, $\mu(D_{2l})+\mu(G_{2l+1})\leqq\mu(G_{2l})$, $\mu(D_{2l+1})+\mu(G_{2l+2})\leqq\mu(G_{2l+1})$ and $\mu\left(K\setminus\bigcup_{j=1}^{t}\left(\bigcup_{-\infty}^{\infty}T^nD_j\right)\right)>\mu(K)-2m\left\{\sum_{j=1}^{t}\mu(D_j)\right\}>\mu(K)-2m\left\{\sum_{j=1}^{t-1}\mu(D_j)+\mu(G_t)\right\}>\ldots>\mu(K)-2m\mu(G)>0$ for any positive integer t. Let $E=G_1\setminus\bigcup_{j=1}^{\infty}(G_j\setminus G_{j+1})$, then $\left(\bigcup_{-\infty}^{\infty}T^nE\right)\cap K=\phi$ (a.e.). Now, let $F=\left\{\bigcup_{j=1}^{\infty}\left(\bigcup_{-\infty}^{\infty}T^nD_j\right)\right\}\cup\left\{\bigcup_{-\infty}^{\infty}T^nE\right\}$, then clearly $F\supset\bigcup_{-\infty}^{\infty}T^nG$ and $\mu(K\setminus F)>\mu(K)-2m\mu(G)>0$. It contradicts to T-osmoticity of A.

It is easy to see from Lemma 1 and Theorem 1 that if T is ergodic then $\sum_{n=1}^{\infty}\rho_n^{-1}=\infty$ [3, 7].

Uniformity of mixing transformations. Now we cosider the converse of Theorem 1. In the mixing relation(iii) if there is a sufficiently large integer N depends only $H\in\mathfrak{K}$ and not depends any choice of A and B such that

$|\rho_n\mu(A\cap T^{-n}B)-\mu(A)\mu(B)|<\varepsilon$ for any number $\varepsilon>0$ and for any integer $n\geqq N$. Then it is called that T has uniformity.

THEOREM 2. *Mixing transformation having uniformity is always ergodic.*

PROOF. Suppose T is not ergodic, i.e., there is non-trivial T-invariant $\mathfrak{B}$-set A. For the dilution sequence $\{\rho_n\}$ we can put a sequence $\{\varepsilon_n\}$ such that $\lim_{n\to\infty}\rho_n\varepsilon_n=0$ (for example let $\varepsilon_n=\frac{1}{n}\rho_n^{-1}$). By the metrically denseness of $\mathfrak{K}$ in $\mathfrak{B}$ there is a $\mathfrak{K}$-set $A\varepsilon_n$ such that $(A+A\varepsilon_n)$

$< \varepsilon_n$; where $A+B$ denotes $(A\setminus B)\cup(B\setminus A)$. Let $B=X\setminus A$, $A_n=H\cap A\varepsilon_n$, $B_n=H\setminus A_n$, $A_{1,n}=A\cap A_n$, $A_{2,n}=B\cap A_n$, $B_{1,n}=B\cap B_n$ and $B_{2,n}=A\cap B_n$, then clearly $\mu(A_{2,n})<\varepsilon_n$ and $\mu(B_{2,n})<\varepsilon_n$. T-invariantness of A follows

$$\mu(A_n\cap T^{-n}B_n)=\mu(A_{1,n}\cap T^{-n}B_{2,n})+\mu(A_{2,n}\cap T^{-n}B_{1,n}).$$

The uniformity of T implies that for any number $\varepsilon>0$ there is a suitable large integer N such that

$$\rho_n\mu(A_n\cap T^{-n}B_n)>\mu(A_n)\mu(B_n)-\varepsilon$$

for any integer $n\geqq N$. Then it is easy to see that

$$\lim_{n\to\infty}\ \rho_n\mu(A_n\cap T^{-n}B_n)>\frac{1}{2}\mu(\hat{A})\mu(\hat{B});\text{ where }\hat{A}=A\cap H\text{ and }\hat{B}=B\cap H.$$

The other hand

$$\rho_n\mu(A_n\cap T^{-n}B_n)=\rho_n\{\mu(A_{1,n}\cap T^{-n}B_{2,n})+\mu(A_{2,n}\cap T^{-n}B_{1,n})\}$$
$$\leqq\rho_n\{\mu(B_{2,n})+\mu(A_{2,n})\}<2\rho_n\varepsilon_n.$$

It shows that $\lim_{n-\infty}\ \rho_n\mu(A_n\cap T^{-n}B_n)=0$. Thus $\mu(\hat{A})\mu(\hat{B})=0$, i.e., $\mu(H\cap A)=0$ or $\mu(H\cap B)=0$ for any $H\in\mathfrak{K}$ of the mixing property(ii). Consequently, $\mu(A)=0$ or $\mu(B)=0$. It is a contradiction.

Theorem 2 shows that it is necessarly some sense of uniformity of mixing transformation T on account of the relation $\sum_{n=1}^{\infty}\rho_n^{-1}=\infty$ implies ergodicity of T. Thus we introduce a weak sense of uniformity of mixing transformations.

Lemma 4. *Let T be an invertible μ-continuous transformation on X. If $A\in\mathfrak{K}$, then $T^{-1}A\in\mathfrak{K}$.*

Proof. If $T^{-1}A$ not belongs to $\mathfrak{K}$, i.e., $T^{-1}A$ has μ-non-null boundary C. Then, for any $b\in C$ and any neighbourhood $U(b)$ of b $U(b)\cap T^{-1}A\neq\phi$ and $U(b)\cap(X\setminus T^{-1}A)\neq\phi$. Continuity of T follows that for any neighbourhood $V(Tb)$ of Tb there is a neighbourhood $U(b)$ of b such that $T(U(b))\subset V(Tb)$. Since $U(b)\cap T^{-1}A\neq\phi$ and $U(b)\cap(X\setminus T^{-1}A)\neq\phi$, hence $T(U(b))\cap A\neq\phi$ and $T(U(b))\cap(X\setminus A)\neq\phi$. Thus, $V(Tb)\cap A\neq\phi$ and $V(Tb)\cap(X\setminus A)\neq\phi$, i.e., Tb is a boundary point of A. Therefore TC is contained in the boundary of A. It contradicts to $A\in\mathfrak{K}$.

Let A and B are both $\mathfrak{K}$-subset of $H\in\mathfrak{K}$ in the mixing property(ii). The mixing reation (iii) shows that for any number $\varepsilon>0$ there is a positive integer N such that $|\rho_n\mu(A\cap T^{-n}B)-\mu(A)\mu(B)|<\varepsilon$ for any integer $n\geqq N$. Moreover, if for any positive integer k and any integer $n\geqq N$,

$$\left|\rho_n\mu\left(\left(\bigcap_{i=0}^{k-1}T^{-i}A\setminus T^{-k}A\right)\bigcap T^{-n}B\right)-\mu\left(\bigcap_{i=0}^{k-1}T^{-i}A\setminus T^{-k}A\right)\mu(B)\right|<\varepsilon\mu\left(\bigcap_{i=0}^{k-1}T^{-i}A\setminus T^{-k}A\right)/\mu(A)$$

holds, then it is called that T has *sequential uniformity*. In this situation we can show that the converse of Theorem 1 is realized as follows.

Lemma 5. *Let T be a mixing transformation, then there is no non-trivial T-invariant $\mathfrak{K}$-set.*

Proof. Let A be a non-trivial T-invariant $\mathfrak{K}$-set. Let $B=X\setminus A$, then clearly $\mu(A)\neq 0$ and $\mu(B)\neq 0$. We can put $H\in\mathfrak{K}$ in the property(ii) of mixing such that $\mu(A_0)\neq 0$ and $\mu(B_0)\neq 0$; where $A_0=H\cap A$ and $B_0=H\cap B$. By the mixing relaion(iii) of T we have $\lim_{n-\infty}\ \rho_n\mu$

$(A_0 \cap T^{-n} B_0) = \mu(A_0)\mu(B_0)$. Since $\mu(A_0 \cap T^{-n} B_0) = 0$ for any positive integer n, hence $\mu(A_0)\mu(B_0) = 0$, i.e., $\mu(A_0) = 0$ or $\mu(B_0) = 0$. It is a contradiction.

THEOREM 3. *Let T be an invertible mixing transformation governed by $\{\rho_n\}$. If T has sequential uniformity and satisfies the condition $\sum_{n=1}^{\infty} \rho_n^{-1} = \infty$, then there is a T-osmotic set.*

PROOE. Suppose that there is no T-osmotic set. From Lemma 1 T is not ergodic, hence there is a non-trivial T-invariant $\mathfrak{B}$-set $\hat{A}$. Lemma 5 implies that $A \notin \mathfrak{R}$. Let $\hat{B} = X \setminus \hat{A}$. Metrically denseness of $\mathfrak{R}$ in $\mathfrak{B}$ follows that there is $\hat{A}_0 \in \mathfrak{R}$ such that $\mu(\hat{A} + \hat{A}_0) < \delta$ for any $\delta > 0$. Let $\hat{B}_0 = X \setminus \hat{A}_0$, $\hat{A}_1 = \hat{A} \cap \hat{A}_0$, $\hat{A}_2 = \hat{B} \cap \hat{A}_0$, $\hat{B}_1 = \hat{B} \cap \hat{B}_0$, $\hat{B}_2 = \hat{A} \cap \hat{B}_0$, then it is clear that $\mu(\hat{A}_2) < \delta$, $\mu(\hat{B}_2) < \delta$. Since $\hat{A} \notin \mathfrak{R}$, hence $\mu(\hat{A}_2) \neq 0$, $\mu(\hat{B}_2) \neq 0$. By the assumption that T-invariant set $\hat{A}$ is not T-osmtic, hence $\bigcup_{-\infty}^{\infty} T^n \hat{B}_2$ is non-trivial T-invariant subst of $\hat{A}$, i.e., let $\hat{C} = \hat{A} \setminus \bigcup_{-\infty}^{\infty} T^n \hat{B}_2$, then $\mu(\hat{C}) \neq 0$. There is $H \in \mathfrak{R}$ in the property(ii) of mixing such that, let $A = \hat{A} \cap H$, $B = \hat{B} \cap H$, $A_0 = \hat{A}_0 \cap H$, $B_0 = \hat{B}_0 \cap H$, $A_1 = \hat{A}_1 \cap H$, $A_2 = \hat{A}_2 \cap H$, $B_1 = \hat{B}_1 \cap H$, $B_2 = \hat{B}_2 \cap H$ and $C = \hat{C} \cap H$, then $\mu(A)$, $\mu(B)$, $\mu(A_0)$, $\mu(B_0)$, $\mu(A_1)$, $\mu(A_2)$, $\mu(B_1)$, $\mu(B_2)$ and $\mu(C)$ are all non-null.

$A_0 \cap T^{-1} A_0 = (A_1 \setminus T^{-1} B_2) \cup (A_2 \cap T^{-1} A_2) \supset A_1 \setminus T^{-1} B_2 = A \setminus B_2 \cap T^{-1} B_2 \supset C$ shows that $\mu(A_0 \cap T^{-1} A_0) \neq 0$. Similarly, $\mu(A_0 \setminus T^{-1} A) \neq 0$. For, if $\mu(A_0 \setminus T^{-1} A_0) = 0$, then $A_0 \subset T^{-1} A_0$ (a.e.). Since T is measure preserving, hence $A_0 = T^{-1} A_0$ (a.e.) i.e., A_0 is a non-trivial T-invariant $\mathfrak{R}$-set. By Lemma 5 it is impossible. Inductively, if $\mu\left(\bigcap_{i=0}^{k} T^{-i} A_0\right) \neq 0$ and $\mu\left(\bigcap_{i=0}^{k=1} T^{-i} A_0 \setminus T^{-k} A_0\right) \neq 0$ for an integer $k > 1$, then, since $\bigcap_{i=0}^{k+1} T^{-i} A_0 \supset A_1 \setminus \bigcup_{j=1}^{k+1} T^{-j} B_2 \supset C$, hence $\mu\left(\bigcap_{j=0}^{k+1} T^{-i} A\right) \neq 0$. Similarly, $\mu\left(\left(\bigcap_{i=0}^{k} T^{-i} A_0\right) \setminus T^{-(k+1)} A_0\right) \neq 0$. For, if $\mu\left(\bigcap_{i=0}^{k} T^{-i} A_0 \setminus T^{-(k+1)} A_0\right) = 0$, then $\bigcap_{i=0}^{k} T^{-i} A_0 \subset \bigcap_{i=0}^{k+1} T^{-i} A$. Consequently, $\bigcap_{i=0}^{k} T^{-i} A_0 = \bigcap_{i=0}^{k+1} T^{-i} A_0$. Thus $T^{-1}\left(\bigcap_{i=0}^{k} T^{-i} A_0\right) = \bigcap_{i=1}^{k+1} T^{-i} A_0 \supset \bigcap_{i=0}^{k+1} T^{-i} A_0$. It follows that $T^{-1}\left(\bigcap_{i=0}^{k} T^{-i} A_0\right) = \bigcap_{i=0}^{k} T^{-i} A_0$, i.e., $\bigcap_{i=0}^{k} T^{-i} A_0$ is a non-trivial T-invariant $\mathfrak{R}$-set. It impossible by Lemma 5.

Now, from the mixing relation(iii) of T there is a positive integer N for any number $\varepsilon > 0$ such that

$$\left|\rho_n \mu(A_0 \cap T^{-n} B_0) - \mu(A_0)\mu(B_0)\right| < \varepsilon \quad \text{for any integer} \quad n \geqq N.$$

Sequential uniformity of T follows,

$$\rho_{N+k}\, \mu\left(\left(\bigcap_{i=0}^{k-1} T^{-i} A_0 \setminus T^{-k} A_0\right) \cap T^{-(N+k)} B_0\right) < \mu\left(\bigcap_{i=0}^{k-1} T^{-i} A_0 \cap T^{-k} A_0\right) \mu(B_0) +$$
$$\varepsilon \mu\left(\bigcap_{i=0}^{k-1} T^{-i} A_0 \setminus T^{-k} A_0\right) / \mu(A_0)$$

for any positive integer k. Since

$$\mu\Big(A_0\cap T^{-(N+k)}B_0\Big)=\mu\Big(\Big(\bigcap_{i=0}^{k}T^{-i}A_0\Big)\cap T^{-(N+k)}B_0\Big)+\sum_{j=1}^{k-1}\mu\Big(\Big(\bigcap_{i=0}^{j-1}T^{-i}A_0\setminus T^{-j}A_0\Big)\cap T^{-(N+k)}B_0\Big)$$

and

$$\sum_{i=1}^{k-1}\rho_{N+k}\,\mu\Big(\Big(\bigcap_{i=0}^{j-1}T^{-i}A_0\setminus T^{-j}A_0\Big)\cap T^{-(N+k)}B_0\Big)<\sum_{j=1}^{k}\mu\Big(\bigcap_{i=0}^{j-1}T^{-i}A_0\setminus T^{-i}A_0\Big)\mu(B_0)+$$

$$\varepsilon\sum_{j=1}^{k}\mu\Big(\bigcap_{i=0}^{j-1}T^{-i}A_0\setminus T^{-j}A_0\Big)/\,\mu(A_0)<\sum_{j=1}^{k}\mu\Big(\bigcap_{i=0}^{j-1}T^{-i}A_0\setminus T^{-j}A_0\Big)+\varepsilon.$$

Therefore,

$$\rho_{N+k}\,\mu\Big(\Big(\bigcap_{i=0}^{k}T^{-i}A_0\Big)\cap T^{-(N+k)}B_0\Big)>\mu(A_0)\mu(B_0)-\sum_{j=1}^{k}\mu\Big(\bigcap_{i=0}^{j-1}T^{-i}A_0\setminus T^{-i}A_0\Big)\mu(B_0)-2\varepsilon.$$

Since $\bigcap_{i=0}^{j-1}T^{-i}A_0\setminus T^{-i}A_0=\Big(\bigcap_{i=0}^{j-1}T^{-i}A_2\setminus T^{-j}A_2\Big)\cup\Big(T^{-j}B_2\setminus\bigcup_{i=0}^{j-1}T^{-i}B_2\Big)$,

hence

$$\sum_{j=1}^{k}\mu\Big(\bigcap_{i=1}^{j-1}T^{-i}A_0\setminus T^{-j}A_0\Big)=\sum_{j=1}^{k}\mu\Big(\bigcap_{i=0}^{j-1}T^{-i}A_0\setminus T^{-j}A_0\Big)+\sum_{j=1}^{k}\mu\Big(T^{-i}B_2\setminus\bigcup_{i=0}^{j-1}T^{-i}B_2\Big)$$

$$<\mu(A_2)+\mu\Big(\bigcup_{i=1}^{\infty}T^{-i}B_2\Big)<\mu(A_2)+\mu(A_1)-\mu(C)=\mu(A_0)-\mu(C).$$

Consequently,

$$\rho_{N+k}\,\mu\Big(\Big(\bigcap_{i=0}^{k}T^{-i}A_0\Big)\cap T^{-(N+k)}B_0\Big)>\mu(A_0)\mu(B_0)-\{\mu(A_0)-\mu(C)\}\mu(B_0)-2\varepsilon$$

$=\mu(C)\mu(B_0)-2\varepsilon>0$ for sufficiently large integer N.

Since

$$\mu(T^{-(N+k)}A_0\cap T^{-(N+k)}B_0)=\mu(A_0\cap B_0)=0,$$

hence

$$\mu\Big(\Big(\bigcap_{i=0}^{k}T^{-i}A_0\Big)\cap T^{-(N+k)}B_0\Big)=\mu\Big(\Big(\bigcap_{i=0}^{k}T^{-i}A_0\setminus T^{-(N+k)}A_0\Big)\cap T^{-(N+k)}B_0\Big).$$

Therefore,

$\rho_{N+k}\,\mu\Big(\bigcap_{i=0}^{k}T^{-i}A_0\setminus T^{-(N+k)}B_0\Big)>\mu(C)\mu(B_0)-2\varepsilon>0$, i.e.,

$$\rho_{N+k}^{-1}<\mu\Big(\bigcap_{i=0}^{k}T^{-i}A_0\setminus T^{-(N+k)}B_0\Big)\Big/\{\mu(C)\mu(B_0)-2\varepsilon\}.$$

It follows that

$$\sum_{k=1}^{\infty}\rho_{N+k}^{-1}<\sum_{k=1}^{\infty}\mu\Big(\bigcap_{i=0}^{k}T^{-i}A_0\setminus T^{-(N+k)}A_0\Big)/\{\mu(C)\mu(B_0)-2\varepsilon\}$$
$$<N\mu(A_0)/\{\mu(C)\mu(B_0)-2\varepsilon\}.$$

It contradicts to $\sum_{i=1}^{\infty}\rho_n^{-1}=\infty$.

Theorem 3 , lemma 2 and lemma 3 follows that

COROLLARY. *If a mixing transformation T has sequential uniformity and T-transitivity, then $\sum_{i=1}^{\infty}\rho_n^{-1}=\infty$ implies that ergodicity and conservativity of T.*

For example, it is easy to see that the mixing transformation defined by shift of measure theoretical sample space constructed by an irreducible and aperiodic Markov chain admitted a non-trivial stationary measure has the sequential uniformity and T-transitivity [1, 3, 7].

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