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<td>タイトル</td>
<td>インフィニット・メジャー上のリピーブラーの同形性についての研究</td>
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<td>作者</td>
<td>TOMATSU, Shizuo</td>
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<tr>
<td>引用</td>
<td>岐阜大学教養部研究報告 vol.17  p.43-49</td>
</tr>
<tr>
<td>発行日</td>
<td>1981</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/20.500.12099/47503">http://hdl.handle.net/20.500.12099/47503</a></td>
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Uniformity of mixing transformations with infinite measure

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(Received Oct. 5, 1981)

Introduction. Mixing transformations on a completely regular topological measure space with infinite measure are not necessarily ergodic (7). The purpose of this note is to treat that some attached conditions of mixing transformations in order to imply the ergodicity.

Let \((X, \mathcal{B}, \mu)\) be a topological measure space of a completely regular topological space \(X\), the \(\sigma\)-field \(\mathcal{B}\) of Borel sets of \(X\), and a measure \(\mu\) on \(\mathcal{B}\), where the measure \(\mu\) assumed to be non-negative, \(\sigma\)-additive, locally finite and tight in the sense that \(\mu(A) = \sup \{\mu(K); K \subset A, K\text{ is compact}\}\) for every \(A \in \mathcal{B}\). A set \(A \in \mathcal{B}\) will be almost clopen if it characteristic function is \(\mu\)-almost everywhere (in what follows it denotes abbreviated a.e.) continuous. Let \(\mathfrak{H}\) be the set of all clopen sets of \(\mathcal{B}\). It is clear that \(\mathfrak{H}\) is a field but not \(\sigma\)-field. It is known that \(\mathfrak{H}\) is \(\mu\)-metrically dense in \(\mathcal{B}\) (7).

An endomorphism \(T\) of \(X\) is called mixing if it has the following properties (6, 7).

(i) \(T\) is continuous (a.e.), and measure preserving.

(ii) There is a sequence of \(\mathfrak{H}\)-subsets \(H_k \in \mathfrak{H}\) \((k = 1, 2, 3, \ldots)\) of \(X\) with finite measure such that \(X = \bigcup_{k=1}^{\infty} H_k\) (a.e.).

(iii) There is a sequence of positive numbers \(\rho_n \ (n = 1, 2, 3, \ldots)\) such that the mixing relation \(\lim_{n \to \infty} \rho_n \mu(A \cap T^{-n}B) = \mu(A) \mu(B)\) holds for any \(\mathfrak{H}\)-sets \(A\) and \(B\) included in some \(H_k\).

\(T\)-invariant subsets. Let \(T\) be a measure preserving transformation on \(X\). A \(T\)-invariant \(\mathcal{B}\)-set \(A\) which has a \(\mathcal{B}\)-subset \(D\) such that \(\bigcup_{n=0}^{\infty} T^n D = A\) (a.e.) and \(0 < \mu(D) < \epsilon\) for any small positive number \(\epsilon\). Such a set \(A\) is called \(T\)-osmotic and the set \(D\) is called generator of \(A\). A \(T\)-invariant \(\mathcal{B}\)-set is called \(T\)-conservative if it has no wandering set. A \(T\)-invariant \(\mathcal{B}\)-set is called \(T\)-ergodic if it has no non-trivial \(T\)-invariant subset. A \(\mathcal{B}\)-set \(D\) is called \(T\)-transitive if for any \(\mu\)-non-null subsets \(E\) and \(F\) of \(D\) there is an integer \(n\) such that \(\mu(E \cap T^{-n}F) \neq 0\). Then it is clear that the followings.

Lemma 1. \(T\)-ergodic set is also \(T\)-osmotic.

Lemma 2. \(T\)-osmotic set which has \(T\)-transitive generator is \(T\)-ergodic.
Proof. Let $A$ be a $T$-osmotic set having $T$-transitive generator $D$. If $A$ is not $T$-ergodic, i.e., $A$ has non-trivial $T$-invariant $\mathcal{B}$-subset $B$. Let $C=A\setminus B$ ($A\setminus B$ denotes the set of elements which belong to $A$ but not $B$), then $\mu(C)\neq 0$. Let $D_1=D\cap B$ and $D_2=D\cap C$, then clearly $\mu(D_1)\neq 0$, $\mu(D_2)\neq 0$, and $D_1$, $D_2$ are generators of $B$, $C$ respectively. Since $D$ is $T$-transitive, hence there is a positive integer $n$ such that $\mu(D_1 \cap T^{-n}D_2)\neq 0$, i.e., $\mu(B\cap C)\neq 0$. It is a contradiction.

Lemma 3. $T$-osmotic set is also $T$-conservative.

Proof. Let $A$ be not $T$-conservative, and its wandering set denotes $\{T^nE\}; n=0, \pm 1, \pm 2, \ldots$ (where $T^nE=E$). Now we put any $\mathcal{B}$-subset $F$ of $A$ such that $0<\mu(F)<\mu(E)$. Let $F\cap T^nE=F_n$, then clearly $F_n$'s are mutually disjoint and $T^{-n}F_n \subset E$. Since $\mu\left(\bigcup_{n=0}^{\infty} T^{-n}F_n\right)\leq \sum_{n=0}^{\infty} \mu(F_n)=\mu(F)<\mu(E)$, hence we can put a $\mathcal{B}$-set $G$ such that $G \subset E\setminus \bigcup_{n=0}^{\infty} T^{-n}F_n$. Then we have

$$T^mG \subset T^mE \setminus \bigcup_{n=0}^{\infty} T^{-n}F_n \subset T^mE \setminus T^m(T^{-n}F_n) = T^mE \setminus F_m = T^mE \setminus F$$

for any integer $m$, i.e., $(T^mG)\cap F=\emptyset$ or equivalently $G \cap (T^mF)=\emptyset$ (\emptyset denotes empty set). It implies that $\bigcup_{n=0}^{\infty} T^nF$ is non-trivial $T$-invariant subset of $A$. This shows that $A$ is not $T$-osmotic.

Mixing transformations. The sequence $\{\rho_n\}$ of positive numbers in the definition of mixing property(iii) is considered as some sort of dilution factor and asymptotically independent of the choice of the $H_n \in \mathcal{G}$ in the definition of mixing property(ii) (6). It is known that the behaviour of the dilution sequence $\{\rho_n\}$ of mixing transformation concerned to ergodicity of the transformation (6, 7).

Now we show the followings with regard to the dilution sequence of mixing transformations.

Theorem 1. Let $T$ be an invertible mixing transformation governed by the sequence $\{\rho_n\}$. If there exist $T$-osmotic set, then $\sum_{n=1}^{\infty} \rho_n^{-1} = \infty$.

Proof. Let $A$ be a $T$-osmotic set, then there is a $H \in \mathcal{G}$ in the definition of mixing property(ii) such that $\mu(H \cap A) \neq 0$. From mixing relation(iii) there is a sufficiently large integer $N$ such that $\rho_n^{-1} > \mu(H \cap T^{-n}H)/2\mu(H)^2$ for any integer $n \geq N$. Now, if we suppose $\sum_{n=1}^{\infty} \rho_n^{-1} < \infty$, then there is an integer $m \geq N$ such that $\sum_{n=m}^{\infty} \mu(H \cap T^{-n}H) < \epsilon$ for any positive number $\epsilon$. Let $\epsilon < \frac{1}{2} \mu(H \cap A)$ and $K=(H \cap A) \setminus \bigcup_{n=m}^{\infty} (H \cap T^{-n}H)$, then $\mu(K) > 0$ and $K \cap T^{-n}K = \emptyset$ for any integer $n \geq m$. Consequently, $K \cap T^{-m}K = \emptyset$ for $n \geq m$. Let $G$ be any $\mathcal{B}$-subset of $A$ such that $0<\mu(G)<\frac{1}{2m} \mu(K)$, if $G \subset K$, then $\mu\left(\bigcup_{n=0}^{\infty} T^nG \cap K\right) < \mu(K)$, i.e., $\mu\left(A \setminus \bigcup_{n=0}^{\infty} T^nG\right) \neq 0$. It contradicts to $T$-osmoticity of $A$. Hence $G$ is not contained in $K$. Let $D_1=G \cap K$ and $G_1=G \cap \left(\bigcup_{n=0}^{\infty} H \cap T^{-n}H\right)$, then $\mu(G_1) \neq 0$. Since $\mu\left(K \setminus \bigcup_{n=0}^{\infty} T^nD_1\right) > \mu(K)-2m\mu(D_1)$
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$\mu(K-2m \mu(G) > 0$, hence $K \setminus \bigcup_{n=1}^{\infty} T^n D_1 \neq \phi(a.e.)$. Let $G_2 = G_1 \setminus \bigcup_{n=1}^{\infty} T^n D_1$. If $\mu(G_2) = 0$, then $G \subset \bigcup_{n=1}^{\infty} T^n D_1 (a.e.)$, i.e., $\bigcup_{n=1}^{\infty} T^n G = \bigcup_{n=1}^{\infty} T^n D_1$. Since $A \setminus \bigcup_{n=1}^{\infty} T^n D_1 \supset K \setminus \bigcup_{n=1}^{\infty} T^n D_1 \neq \phi(a.e.)$, hence $A \setminus \bigcup_{n=1}^{\infty} T^n G(a.e.)$. It contradicts to $T$-osmoticity of $A$. Thus $\mu(G_2) \neq 0$. Let $D_2 = K \cap T^{-1} G_2$, then

$$\mu(K \setminus \bigcup_{n=1}^{\infty} T^n D_1) > \mu(K) - 2m \{ \mu(D_1) + \mu(D_2) \} > \mu(K) - 2m \mu(G) > 0.$$ Let $G_3 = G_2 \setminus \bigcup_{n=1}^{\infty} T^n D_2$. Then $TD_2 = TK \cap G_3 \subset G_3 \setminus G_2$, thus $\mu(D_2) + \mu(D_3) \leq \mu(G_2)$. If $\mu(G_3) = 0$, then $G \subset \bigcup_{n=1}^{\infty} T^n D_1 \cup \bigcup_{n=1}^{\infty} T^n D_2$. Hence

$$A \setminus \bigcup_{n=1}^{\infty} T^n G \supset A \setminus \bigcup_{n=1}^{\infty} T^n D_1 \supset K \setminus \bigcup_{n=1}^{\infty} T^n D_1 \neq \phi (a.e.).$$

It contradicts to $T$-osmoticity of $A$. Thus $\mu(G_3) \neq 0$. Let $D_3 = K \cap T G_3$, then

$$\mu(K \setminus \bigcup_{n=1}^{\infty} T^n D_1) > \mu(K) - 2m \sum_{j=1}^{3} \mu(D_j) > \mu(K) - 2m (\mu(D_1) + \mu(D_2) + \mu(D_3) + \mu(G_3)) > \mu(K) - 2m \mu(D_1) + \mu(G_3) > 0.$$ Inductively, let $D_{2l} = K \cap T^{-1} G_{2l}, G_{2l+1} = G_{2l} \setminus \bigcup_{n=1}^{\infty} T^n D_{2l}, D_{2l+1} = K \cap TG_{2l+1}, G_{2l+2} = G_{2l+1} \setminus \bigcup_{n=1}^{\infty} T^n D_{2l+1}$, then $\mu(G_{2l+1}) \neq 0$, $\mu(D_{2l+2}) \neq 0$, $\mu(D_{2l}) + \mu(D_{2l+1}) \leq \mu(G_{2l+1})$ and $\mu(K \setminus \bigcup_{n=1}^{\infty} T^n D_{2l}) > \mu(K) - 2m \sum_{j=1}^{t} \mu(D_j) > \mu(K) - 2m \mu(G_{2l+1}) + \mu(D_{2l+1}) \leq \mu(G_{2l+2}) \leq \mu(G_{2l+1})$ and $\mu(K \setminus \bigcup_{n=1}^{\infty} T^n D_{2l}) > \mu(K) - 2m \mu(G_{2l+1}) > \mu(K) - 2m \mu(G) > 0$ for any positive integer $t$.

Let $E = G_1 \setminus \bigcup_{n=1}^{\infty} (G_1 \setminus G_{n+1})$, then $\bigcup_{n=1}^{\infty} T^n E \cap K = \phi (a.e.)$. Now, let $F = \bigcup_{n=1}^{\infty} \bigcup_{n=1}^{\infty} T^n D_j \cup \bigcup_{n=1}^{\infty} T^n E$, then clearly $F \supset \bigcup_{n=1}^{\infty} T^n G$ and $\mu(K \setminus F) > \mu(K) - 2m \mu(G) > 0$. It contradicts to $T$-osmoticity of $A$.

It is easy to see from Lemma 1 and Theorem 1 that if $T$ is ergodic then $\sum_{n=1}^{\infty} \rho_{n-1}^{-1} = \infty (3, 7)$.

**Uniformity of mixing transformations.** Now we consider the converse of Theorem 1. In the mixing relation(iii) if there is a sufficiently large integer $N$ depends only $H \in \mathcal{F}$ and not depends any choice of $A$ and $B$ such that

$$| \rho_n \mu(A \cap T^{-n} B) - \mu(A) \mu(B) | < \epsilon$$

for any number $\epsilon > 0$ and for any integer $n \geq N$. Then it is called that $T$ has uniformity.

**Theorem 2.** Mixing transformation having uniformity is always ergodic.

**Proof.** Suppose $T$ is not ergodic, i.e., there is non-trivial $T$-invariant $\mathcal{F}$-set $A$. For the dilution sequence $\{ \rho_n \}$ we can put a sequence $\{ \epsilon_n \}$ such that $\lim_{n \to \infty} \rho_n \epsilon_n = 0$ (for example let $\epsilon_n = \frac{1}{n} \rho_n^{-1}$). By the metrically denseness of $\mathcal{F}$ in $\mathcal{F}$ there is a $\mathcal{F}$-set $A \epsilon_n$ such that $(A + A \epsilon_n)$
< $\varepsilon_\ast$; where $A + B$ denotes $(A \setminus B) \cup (B \setminus A)$. Let $B = X \setminus A$, $A_0 = H \cap A$, $B_0 = H \setminus A$, $A_{1,n} = A \cap A_n$, $A_{2,n} = B \cap A_n$, $B_{1,n} = B \cap B_n$ and $B_{2,n} = A \cap B_n$, then clearly $\mu(A_{2,n}) < \varepsilon_\ast$ and $\mu(B_{2,n}) < \varepsilon_\ast$. $T$-invariantness of $A$ follows

$$
\mu(A_n \cap T^{-n} B_n) = \mu(A_{1,n} \cap T^{-n} B_{2,n}) + \mu(A_{2,n} \cap T^{-n} B_{1,n}).
$$

The uniformity of $T$ implies that for any number $\varepsilon > 0$ there is a suitable large integer $N$ such that

$$
\rho_n \mu(A_n \cap T^{-n} B_n) > \mu(A_n) \mu(B_n) - \varepsilon
$$

for any integer $n \geq N$. Then it is easy to see that

$$
\lim_{n \to \infty} \rho_n \mu(A_n \cap T^{-n} B_n) < \frac{1}{2} \mu(\hat{A}) \mu(\hat{B});
$$

where $\hat{A} = A \cap H$ and $\hat{B} = B \cap H$.

The other hand

$$
\rho_n \mu(A_n \cap T^{-n} B_n) = \rho_n \{\mu(A_{1,n} \cap T^{-n} B_{2,n}) + \mu(A_{2,n} \cap T^{-n} B_{1,n})\}
\leq \rho_n \{\mu(B_{2,n}) + \mu(A_{2,n})\} < 2 \rho_n \varepsilon_\ast.
$$

It shows that $\lim_{n \to \infty} \rho_n \mu(A_n \cap T^{-n} B_n) = 0$. Thus $\mu(\hat{A}) \mu(\hat{B}) = 0$, i.e., $\mu(H \cap A) = 0$ or $\mu(H \cap B) = 0$ for any $H \in \mathcal{F}$ of the mixing property(ii). Consequently, $\mu(A) = 0$ or $\mu(B) = 0$. It is a contradiction.

Theorem 2 shows that it is necessarily some sense of uniformity of mixing transformation $T$ on account of the relation $\sum_{n=1}^{\infty} \rho_n^{-1} < \infty$ implies ergodicity of $T$. Thus we introduce a weak sense of uniformity of mixing transformations.

**Lemma 4.** Let $T$ be an invertible $\mu$-continuous transformation on $X$. If $A \in \mathcal{F}$, then $T^{-1} A \in \mathcal{F}$.

**Proof.** If $T^{-1} A$ not belongs to $\mathcal{F}$, i.e., $T^{-1} A$ has $\mu$-non-null boundary $C$. Then, for any $b \in C$ and any neighbourhood $U(b)$ of $b$ $U(b) \cap T^{-1} A \neq \phi$ and $U(b) \cap (X \setminus T^{-1} A) \neq \phi$. Continuity of $T$ follows that for any neighbourhood $V(Tb)$ of $Tb$ there is a neighbourhood $U(b)$ of $b$ such that $T(U(b)) \subset V(Tb)$. Since $U(b) \cap T^{-1} A \neq \phi$ and $U(b) \cap (X \setminus T^{-1} A) \neq \phi$, hence $T(U(b)) \cap A \neq \phi$ and $T(U(b)) \cap (X \setminus A) \neq \phi$. Thus, $V(Tb) \cap A \neq \phi$ and $V(Tb) \cap (X \setminus A) \neq \phi$, i.e., $Tb$ is a boundary point of $A$. Therefore $TC$ is contained in the boundary of $A$. It contradicts to $A \in \mathcal{F}$.

Let $A$ and $B$ are both $\mathcal{F}$-subset of $H \in \mathcal{F}$ in the mixing property(ii). The mixing relation (iii) shows that for any number $\varepsilon > 0$ there is a positive integer $N$ such that $|\rho_n \mu(A \cap T^{-n} B) - \mu(A) \mu(B)| < \varepsilon$ for any integer $n \geq N$. Moreover, if for any positive integer $k$ and any integer $n \geq N$,

$$
|\rho_n \mu(\bigcap_{i=0}^{k-1} T^{-i} A \setminus T^{-k} A) \cap T^{-n} B) - \mu(\bigcap_{i=0}^{k-1} T^{-i} A \setminus T^{-k} A) \mu(B)| < \varepsilon \mu(\bigcap_{i=0}^{k-1} T^{-i} A \setminus T^{-k} A) / \mu(A)
$$

holds, then it is called that $T$ has sequential uniformity. In this situation we can show that the converse of Theorem 1 is realized as follows.

**Lemma 5.** Let $T$ be a mixing transformation, then there is no non-trivial $T$-invariant $\mathcal{F}$-set.

**Proof.** Let $A$ be a non-trivial $T$-invariant $\mathcal{F}$-set. Let $B = X \setminus A$, then clearly $\mu(A) \neq 0$ and $\mu(B) \neq 0$. We can put $H \in \mathcal{F}$ in the property(ii) of mixing such that $\mu(A_0) \neq 0$ and $\mu(B_0) \neq 0$; where $A_0 = H \cap A$ and $B_0 = H \cap B$. By the mixing relation(iii) of $T$ we have $\lim_{n \to \infty} \rho_n \mu \mu(A_0 \cap T^{-n} B_0)$
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$(A_0 \cap T^{-n} B_0) = \mu(A_0) \mu(B_0)$. Since $\mu(A_0 \cap T^{-n} B_0) = 0$ for any positive integer $n$, hence $\mu(A_0) \mu(B_0) = 0$, i.e., $\mu(A_0) = 0$ or $\mu(B_0) = 0$. It is a contradiction.

**Theorem 3.** Let $T$ be an invertible mixing transformation governed by $\{\rho_n\}$. If $T$ has sequential uniformity and satisfies the condition $\sum_{n=1}^{\infty} \rho_n^{-1} = \infty$, then there is a $T$-osmotic set.

**Proof.** Suppose that there is no $T$-osmotic set. From Lemma 1 $T$ is not ergodic, hence there is a non-trivial $T$-invariant $\mathcal{F}$-set $\hat{A}$. Lemma 5 implies that $A \not\in \mathcal{F}$. Let $\hat{B} = X \setminus \hat{A}$. Metrically denseness of $\hat{A}$ in $\mathcal{F}$ follows that there is $\hat{A}_0 \in \mathcal{F}$ such that $\mu(\hat{A} + \hat{A}_0) < \delta$ for any $\delta > 0$. Let $\hat{B}_0 = X \setminus \hat{A}_0$, $\hat{A}_1 = \hat{A} \cap \hat{A}_0$, $\hat{A}_2 = \hat{A} \cap \hat{B}_0$, $\hat{B}_1 = \hat{B} \cap \hat{B}_0$, $\hat{B}_2 = \hat{A} \cap \hat{B}_0$, then it is clear that $\mu(\hat{A}_2) < \delta$, $\mu(\hat{B}_2) < \delta$. Since $\hat{A} \not\in \mathcal{F}$, hence $\mu(\hat{A}_2) \neq 0$, $\mu(\hat{B}_2) \neq 0$. By the assumption that $T$-invariant set $\hat{A}$ is not $T$-osmotic, hence $\hat{C} = \hat{A} \setminus \bigcup_{n=0}^{\infty} T^n \hat{B}_2$, then $\mu(\hat{C}) \neq 0$. There is $H \in \mathcal{F}$ in the property(ii) of mixing such that, let $A = \hat{A} \cap H$, $B = \hat{B} \cap H$, $A_0 = \hat{A}_0 \cap H$, $B_0 = \hat{B}_0 \cap H$, $A_1 = \hat{A}_1 \cap H$, $A_2 = \hat{A}_2 \cap H$, $B_1 = \hat{B}_1 \cap H$, $B_2 = \hat{B}_2 \cap H$ and $C = \hat{C} \cap H$, then $\mu(A)$, $\mu(B)$, $\mu(A_0)$, $\mu(B_0)$, $\mu(A_1)$, $\mu(A_2)$, $\mu(B_1)$, $\mu(B_2)$ and $\mu(C)$ are all non-null. $A_0 \cap T^{-1} A_0 = (A_1 \setminus T^{-1} B_2) \supseteq (A_2 \setminus T^{-1} B_2) \supseteq C \subseteq A_1 \setminus T^{-1} B_2 \supseteq A_2 \setminus T^{-1} B_2 \supseteq C$ shows that $\mu(A_0 \cap T^{-1} A_0) \neq 0$. Similarly, $\mu(A_0 \setminus T^{-1} A) \neq 0$. For, if $\mu(A_0 \setminus T^{-1} A_0) = 0$, then $A_0 \subset T^{-1} A_0$ (a.e.). Since $T$ is measure preserving, hence $A_0 = T^{-1} A_0$ (a.e.) i.e., $A_0$ is a non-trivial $T$-invariant $\mathcal{F}$-set. By Lemma 5 it is impossible. Inductively, if $\mu\left(\bigcap_{i=0}^{k} T^{-i} A_0 \setminus T^{-k} A_0\right) \neq 0$ for an integer $k > 1$, then, $\bigcap_{i=0}^{k+1} T^{-i} A_0 \supseteq A_1 \setminus \bigcup_{j=1}^{k+1} T^{-j} B_2 \supseteq C$, hence $\mu\left(\bigcap_{i=0}^{k+1} T^{-i} A\right) \neq 0$. Similarly, $\mu\left(\bigcap_{i=0}^{k+1} T^{-i} A_0 \setminus T^{-(k+1)} A_0\right) \neq 0$. For, if $\mu\left(\bigcap_{i=0}^{k+1} T^{-i} A_0 \setminus T^{-(k+1)} A_0\right) = 0$, then $\bigcap_{i=0}^{k+1} T^{-i} A \supseteq \bigcap_{i=0}^{k+1} T^{-i} A_0$. Consequently, $\bigcap_{i=0}^{k+1} T^{-i} A_0 = \bigcap_{i=0}^{k+1} T^{-i} A_0$. Thus $T^{-(k+1)} \left(\bigcap_{i=0}^{k+1} T^{-i} A_0\right) = \bigcap_{i=0}^{k+1} T^{-i} A$. It follows that $T^{-(k+1)} \left(\bigcap_{i=0}^{k+1} T^{-i} A_0\right) = \bigcap_{i=0}^{k+1} T^{-i} A$, i.e., $\bigcap_{i=0}^{k+1} T^{-i} A_0$ is a non-trivial $T$-invariant $\mathcal{F}$-set. It impossible by Lemma 5.

Now, from the mixing relation(iii) of $T$ there is a positive integer $N$ for any number $\varepsilon > 0$ such that

$$\left|\rho_{n} \mu(A_0 \cap T^{-n} B_0) - \mu(A_0) \mu(B_0)\right| < \varepsilon$$

for any integer $n \geq N$.

Sequential uniformity of $T$ follows,

$$\rho_{n+k} \mu\left(\bigcap_{i=0}^{k-1} T^{-i} A_0 \setminus T^{-k} A_0\right) \cap T^{-(N+k)} B_0 < \mu\left(\bigcap_{i=0}^{k-1} T^{-i} A_0 \cap T^{-k} A_0\right) \mu(B_0) + \varepsilon \mu\left(\bigcap_{i=0}^{k-1} T^{-i} A_0 \setminus T^{-k} A_0\right) / \mu(A_0)$$

for any positive integer $k$. Since
\[
\mu(A_0 \cap T^{-(N+k)}B_0) = \mu\left(\bigcap_{i=0}^{k} T^{-i} A_0 \cap T^{-(N+k)}B_0\right) + \sum_{j=1}^{k-1} \mu\left(\bigcap_{i=0}^{j-1} T^{-i} A_0 \cap T^{-j} A_0 \cap T^{-(N+k)}B_0\right)
\]
and
\[
\sum_{j=1}^{k} \rho_{N+k} \mu\left(\bigcap_{i=0}^{j-1} T^{-i} A_0 \cap T^{-j} A_0 \cap T^{-(N+k)}B_0\right) < \sum_{j=1}^{k} \mu\left(\bigcap_{i=0}^{j-1} T^{-i} A_0 \cap T^{-j} A_0\right) \mu(B_0) + \epsilon \sum_{j=1}^{k} \mu\left(\bigcap_{i=0}^{j-1} T^{-i} A_0 \cap T^{-j} A_0\right) / \mu(A_0) < \sum_{j=1}^{k} \mu\left(\bigcap_{i=0}^{j-1} T^{-i} A_0 \cap T^{-j} A_0\right) + \epsilon.
\]
Therefore,
\[
\rho_{N+k} \mu\left(\bigcap_{i=0}^{k} T^{-i} A_0 \cap T^{-(N+k)}B_0\right) > \mu(A_0) \mu(B_0) - \sum_{j=1}^{k} \mu\left(\bigcap_{i=0}^{j-1} T^{-i} A_0 \cap T^{-j} A_0\right) \mu(B_0) - 2\epsilon.
\]
Since \(\bigcap_{i=0}^{j-1} T^{-i} A_0 \cap T^{-j} A_0 = \bigcap_{i=0}^{j-1} T^{-i} A_2 \cap T^{-j} A_2 \cup \bigcup_{i=0}^{j-1} T^{-i} B_2 \bigcup_{i=0}^{j-1} T^{-i} B_2\),

hence
\[
\sum_{j=1}^{k} \mu\left(\bigcap_{i=0}^{j-1} T^{-i} A_0 \cap T^{-j} A_0\right) = \sum_{j=1}^{k} \mu\left(\bigcap_{i=0}^{j-1} T^{-i} A_0 \cap T^{-j} A_0\right) + \sum_{j=1}^{k} \mu\left(\bigcup_{i=0}^{j-1} T^{-i} B_2 \bigcup_{i=0}^{j-1} T^{-i} B_2\right)
\]
\[
< \mu(A_2) + \mu\left(\bigcup_{i=1}^{m} T^{-i} B_2\right) < \mu(A_2) + \mu(A_2) = \mu(A_2) - \mu(C) = \mu(A_0) - \mu(C).
\]
Consequently,
\[
\rho_{N+k} \mu\left(\bigcap_{i=0}^{k} T^{-i} A_0 \cap T^{-(N+k)}B_0\right) > \mu(A_0) \mu(B_0) - \mu(A_0) - \mu(C) \mu(B_0) - 2\epsilon
\]
\[
= \mu(C) \mu(B_0) - 2\epsilon > 0\] for sufficiently large integer \(N\).

Since \(\mu(T^{-(N+k)}A_0 \cap T^{-(N+k)}B_0) = \mu(A_0 \cap B_0) = 0\),

hence
\[
\mu\left(\bigcap_{i=0}^{k} T^{-i} A_0 \cap T^{-(N+k)}B_0\right) = \mu\left(\bigcap_{i=0}^{k} T^{-i} A_0 \cap T^{-(N+k)}A_0 \cap T^{-(N+k)}B_0\right).
\]
Therefore,
\[
\rho_{N+k} \mu\left(\bigcap_{i=0}^{k} T^{-i} A_0 \cap T^{-(N+k)}B_0\right) > \mu(C) \mu(B_0) - 2\epsilon > 0, i.e.,
\]
\[
\rho_{N+k}^{-1} < \mu\left(\bigcap_{i=0}^{k} T^{-i} A_0 \cap T^{-(N+k)}B_0\right) / \mu(C) \mu(B_0) - 2\epsilon).
\]
It follows that
\[
\sum_{k=1}^{m} \rho_{N+k}^{-1} < \sum_{k=1}^{m} \mu\left(\bigcap_{i=0}^{k} T^{-i} A_0 \cap T^{-(N+k)}A_0\right) / \mu(C) \mu(B_0) - 2\epsilon
\]
\[
< N \mu(A_0) / \mu(C) \mu(B_0) - 2\epsilon\] It contradicts to \(\sum_{k=1}^{m} \rho_{N+k}^{-1} = \infty\).

Theorem 3, lemma 2 and lemma 3 follows that

**Corollary.** If a mixing transformation \(T\) has sequential uniformity and \(T\)-transitivity, then
\(\sum_{i=1}^{m} \rho_{N+k}^{-1} = \infty\) implies that ergodicity and conservativity of \(T\).

For example, it is easy to see that the mixing transformation defined by shift of measure
theoretical sample space constructed by an irreducible and aperiodic Markov chain admitted a non-trivial stationary measure has the sequential uniformity and T-transitivity \((1, 3, 7)\).

**References**