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ON THE JONES POLYNOMIAL OF SYMMETRIC UNIONS WITH TWO COMPONENTS

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Abstract. A symmetric union is a link with a diagram, obtained from diagrams of a knot in the 3-space and its mirror image. In this paper, we give certain formulas of the Jones polynomial of a link with a symmetric union presentation and consider an invariant of symmetric union, which is called the minimal twisting number and show that there exists a link with a symmetric union presentation such that the minimal twisting number is strictly larger than the sum of the minimal twisting numbers of its components.

Key words: symmetric union, Jones polynomial, ribbon link.

1. Introduction

A *symmetric union*, which is a generalized operation of the connected sum of a knot in the 3-space and its mirror image, was first introduced by Kinoshita and Terasaka [5]. They showed that the Alexander polynomial depends only on the parity of the number of half-twists of a trivial tangle on the symmetry axis. Recently, Lamm [6] generalized their definition and considered the relationship between a symmetric union and a ribbon link. (See [3] for the definition.) Every link with a symmetric union presentation is a ribbon link and Lamm showed that every ribbon knot with minimal crossing number ≤ 10 has a symmetric union presentation [6][2] and it is known that all two-bridge ribbon knots can be represented as symmetric unions. [7] [9].

Let $\overline{V}_L(t) = V_L(t)/V_{O^n}(t)$ for an oriented link L of n components, where $V_L(t)$ and $V_{O^n}(t)$ are the Jones polynomial of L and the n-component trivial link O^n respectively. (See Section 2 for the definition.)

It is known that the Alexander polynomial of a symmetric union of n components ($n \geq 2$) is zero [6]. In this paper we study the Jones polynomial of links with symmetric union presentations and its topological properties.

Theorem 1.1. *Let $\overline{L}$ be a link with a symmetric union presentation of the form $D \cup D^*(\infty_2, m)$. Then*
$\overline{V}_{\overline{L}}(t) = (-1)^m t^{-m} \overline{V}_{D\cup D^*(\infty_2,0)}(t) + (1-(-1)^m t^{-m})\overline{V}_{D\cup D^*(\infty_2,\infty)}(t)$.

A link is called *amphicheiral*, if it is isotopic to its mirror image. By Theorem 1.1, we have the following.

Theorem 1.2. *Let $\overline{L}$ be a link with a symmetric union presentation of the form $D \cup D^*(\infty_2, m)$. Then $t^m \overline{V}_{\overline{L}}(t) + (-1)^m \overline{V}_{\overline{L}}(t^{-1}) = (t^m + (-1)^m)\overline{V}_{D\cup D^*(\infty_2,0)}(t)$. In particular, if $\overline{K}$ is amphicheiral, then $\overline{V}_{\overline{K}}(t) = \overline{V}_{D_K\cup D_K^*(\infty_2,0)}(t)$.*

By using the similar proof of Theorem 1.2, we can show that following theorem.

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Theorem 1.3. *Let $\overline{L}$ be a link with a symmetric union presentation of the form $D \cup D^*(\infty_2, m)$. Then $t^m \overline{V}_{\overline{L}}(t) - t^{-m}\overline{V}_{\overline{L}}(t^{-1}) = (t^m - t^{-m})\overline{V}_{D\cup D^*(\infty_2,\infty)}(t)$. In particular, if $\overline{L}$ is amphicheiral, then $\overline{V}_{\overline{L}}(t) = \overline{V}_{D\cup D^*(\infty_2,\infty)}(t)$.*

Now we restrict to the special values of the Jones polynomial. We denote a (Laurent) polynomial $f(t)$ evaluated at r by $[f(t)]_{t=r}$.

Theorem 1.4. *Let $\overline{L}$ be a link with a symmetric union presentation of the form $D \cup D^*(\infty_2, m)$. Then*
$[\frac{d}{dt}\overline{V}_{\overline{L}}(t)]_{t=-1} = m\{\overline{V}_{D\cup D^*(\infty_2,0)}(-1) - \overline{V}_{D\cup D^*(\infty_2,\infty)}(-1)\}$.

Corollary 1.5. *Let $\overline{L}$ be a link with a symmetric union presentation of the form $D \cup D^*(\infty_2, m)$. Then $[\frac{d}{dt}\overline{V}_{\overline{L}}(t)]_{t=-1} \equiv 0 \ mod \ 8|m|$.*

Remark 1.6. These results can be generalized to the case of $D \cup D^*(\infty_\mu, m)$ $(\mu \geq 2)$.

In this paper, all knots and links are oriented unless otherwise stated. In Section 2, we give the definitions of the Jones polynomial and a symmetric union. In Section 3, we shall prove Theorem 1.1 and Theorem 1.2. In Section 4, we shall prove Theorem 1.4 and Corollary 1.5. In Section 5, we introduce the *minimal twisting number* of a link with a symmetric union presentation. It is the smallest number of trivial tangles (with twists) appearing on the axis of a symmetric union presentation of a link, the minimum taken over all symmetric union presentations for the link. We shall show that there exists a symmetric union such that the minimal twisting number is strictly larger than the sum of the minimal twisting numbers of its components.

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2. Definitions

Definition 2.1. Let L be a link in the 3-space. We denote a diagram of L by D_L. The *bracket polynomial* of a diagram of a link L, $< D_L >$ can be defined as a polynomial which satisfies the following identities.

i) $<$ ◯ $>= 1$,

ii) $< D_L \cup$ ◯ $>= -(A^2 + A^{-2}) < D_L >$,

iii) $<$  $>= A <$  $> +A^{-1} <$  $>$.

We defines $V_{D_L}(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ by $V_{D_L}(t) = \{(-A^3)^{-\omega(D_L)} < D_L >\}_{t^{1/2}=A^{-2}}$ for any diagram D_L for L, where ω is the *writhe* of the diagram. (The writhe is the number of positive crossings of D_L minus the number of negative crossings of D_L.) It is shown that $V_{D_L}(t)$ is an invariant of the link [8][4]. Then we denote $V_{D_L}(t)$ by $V_L(t)$ and call it the *Jones polynomial* of L.

Here we define a symmetric union in [6] as follows. We denote the (trivial) tangles made of half twists by integers $n \in \mathbb{Z}$ and the horizontal trivial tangle by ∞ as in Figure 1.

Definition 2.2. Let D be an unoriented link diagram and D^* the diagram D reflected at an axis in the plane. If in the symmetric placement of D and D^* we replace the tangles $T_i = 0, (i = 1, \ldots, k)$ on the symmetry axis by $T_i = \infty$ for $i = 1, \ldots, \mu$ and $T_i = m_i \in \mathbb{Z}$ for $i = \mu + 1, \ldots, k$. We call the result a *symmetric union* of D and D^* and denote it by $D \cup D^*(\infty_\mu, m_{\mu+1}, \ldots, m_k)$. See Figure 1 in the case when $\mu = 1$.

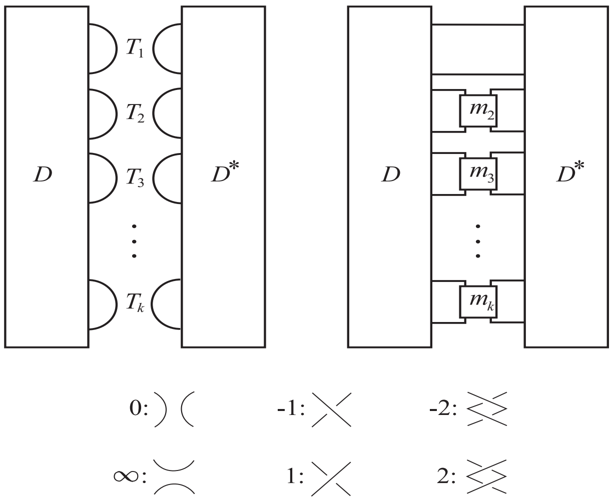


FIGURE 1

If a link L has a diagram $D \cup D^*(\infty_\mu, m_{\mu+1}, \ldots, m_k)$, then the diagram is called a *symmetric union presentation* for L and we say that the link L is a *symmetric union*.

3. THE JONES POLYNOMIAL

Proof of Theorem 1.1. By using a skein relation of Kauffman bracket polynomial, we have

$$< D \cup D^*(\infty_2, m) >= A^{m/|m|} D \cup D^*(\infty_2, m-1) + F_1 < D \cup D^*(\infty_2, \infty) >$$

$$= (A^{m/|m|})^{|m|} < D \cup D^*(\infty_2, 0) > + (F_1 + \cdots + F_{|m|}) < D \cup D^*(\infty_2, \infty) >$$

$$= A^m < D \cup D^*(\infty_2, 0) > + (\textstyle\sum_{j=1}^{|m|} F_j) < D \cup D^*(\infty_2, \infty) >$$

Now we calculate a formula of $\sum_{j=1}^{|m|} F_j$ by considering the unknot instead of K as follows. We assume that D is a diagram as in Figure 2 so that we have a symmetric union of the unknot. We denote the diagram by $D_\circ$.

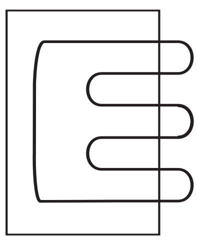


FIGURE 2

Then the resultant symmetric union is a diagram of the unknot with r crossings where $r = |m|$ such that it can be transformed into a diagram of the unknot with no crossings by r type I Reidemeister moves. Thus we have

$$< D_\circ \cup D_\circ^*(\infty_2, m) >= (-A^{-3m/|m|})^{|m|}(-A^{-2} - A^2) = (-1)^{|m|}A^{-3m}(-A^{-2} - A^2),$$

$$< D_\circ \cup D_\circ^*(\infty_2, 0) >= -A^{-2} - A^2,$$

$$< D_\circ \cup D_\circ^*(\infty_2, \infty) >= (A^{-2} + A^2)^2.$$

Then we have

$$\sum_{j=1}^{|m|} F_j = \frac{< D_\circ \cup D_\circ^*(\infty_2, m) > -A^m < D_\circ \cup D_\circ^*(\infty_2, 0) >}{< D_\circ \cup D_\circ^*(\infty_2, \infty) >}$$

$$= \frac{(-1)^{|m|}A^{-3m} - A^m}{-A^{-2} - A^2} = \frac{(-A^{-3})^m - A^m}{-A^{-2} - A^2}.$$

Since $\omega(D \cup D^*(\infty_2, m)) = -m$, we obtain that

$$V_{D\cup D^*(\infty_2,m)}(A) = (-A^3)^m < D \cup D^*(\infty_2, m) >$$

$$= (-A^3)^m \{A^m < D \cup D^*(\infty_2, 0) > + \frac{(-A^{-3})^m - A^m}{-A^{-2} - A^2} < D \cup D^*(\infty_2, \infty) >\}$$

$$= (-1)^m A^{4m} < D \cup D^*(\infty_2, 0) > + \frac{1 - (-1)^m A^{4m}}{-A^{-2} - A^2} < D \cup D^*(\infty_2, \infty) >$$

Using $t^{1/2} = A^{-2}$, we have

$$\overline{V}_{D\cup D^*(\infty_2,m)}(t) = (-1)^m t^{-m}\overline{V}_{D\cup D^*(\infty_2,0)}(t) + (1 - (-1)^m t^{-m})\overline{V}_{D\cup D^*(\infty_2,\infty)}(t).$$

Proof of Theorem 1.2. The first part of the theorem is obtained as follows. By Theorem 1.1, we have

$$t^m\overline{V}_{\overline{K}}(t) + (-1)^m\overline{V}_{\overline{K}}(t^{-1}) = t^m((-1)^m t^{-m}\overline{V}_{D\cup D^*(\infty_2,0)}(t) + (1 - (-1)^m t^{-m})\overline{V}_{D\cup D^*(\infty_2,\infty)}(t))+$$

$$(-1)^m((-1)^m t^m V_{D\cup D^*(\infty_2,0)}(t^{-1}) + (1 - (-1)^m t^m)\overline{V}_{D\cup D^*(\infty_2,\infty)}(t^{-1}))$$

$$= (-1)^m V_{D\cup D^*(\infty_2,0)}(t) + (t^m - (-1)^m)\overline{V}_{D\cup D^*(\infty_2,\infty)}(t))+$$

$$t^m V_{D\cup D^*(\infty_2,0)}(t) + ((-1)^m - t^m)\overline{V}_{D\cup D^*(\infty_2,\infty)}(t)) = (t^m + (-1)^m)V_{D\cup D^*(\infty_2,0)}(t)$$

The latter part of the theorem follow immediately from the first part because $V_{\overline{K}}(t) = V_{\overline{K}}(t^{-1})$ if $\overline{K}$ is amphicheiral ([8], p.29).

4. Evaluation of the derivative at −1

Proof of Theorem 1.4. By Theorem 1.2, we know that

$$\frac{d}{dt}\overline{V}_{\overline{K}}(t) = \frac{d}{dt}((-1)^m t^{-m}\overline{V}_{D\cup D^*(\infty_2,0)}(t)) + \frac{d}{dt}((1 - (-1)^m t^{-m})\overline{V}_{D\cup D^*(\infty_2,\infty)}(t)).$$

Since $[\frac{d}{dt}\overline{V}_{D\cup D^*(\infty_2,0)}(t)]_{t=-1} = 0$, we have

$$[\frac{d}{dt}((-t^{-1})^m\overline{V}_{D\cup D^*(\infty_2,0)}(t))]_{t=-1} = [\frac{d}{dt}(-t^{-1})^m]_{t=-1}(\overline{V}_{D\cup D^*(\infty_2,0)}(-1)) = m(\overline{V}_{D\cup D^*(\infty_2,0)}(-1)).$$

On the one hand, we have

$$\begin{aligned}&[\frac{d}{dt}((1-(-t^{-1})^m)\overline{V}_{D_K\cup D_K^*(\infty_2,\infty)}(t))]_{t=-1}\\ &= -m[\overline{V}_{D\cup D^*(\infty_2,\infty)}(t)]_{t=-1} + [(1-(-t^{-1})^m)\frac{d}{dt}\overline{V}_{D\cup D^*(\infty_2,\infty)}(t)]_{t=-1}\\ &= -m\overline{V}_{D\cup D^*(\infty_2,\infty)}(-1).\end{aligned}$$

Therefore we have

$$[\frac{d}{dt}\overline{V}_{\overline{K}}(t)]_{t=-1} = m\overline{V}_{D\cup D^*(\infty_2,0)}(-1) - m\overline{V}_{D\cup D^*(\infty_2,\infty)}(-1).$$

Here we need the following theorem due to Eisermann.

Theorem 4.1. [1] *If K be a ribbon link, then $\overline{V}_K(-1) \equiv 1$ mod 8.*

Proof of Corollary 1.5. By Theorem 1.4, we have

$$[\frac{d}{dt}\overline{V}_{\overline{K}}(t)]_{t=-1} = m\overline{V}_{D\cup D^*(\infty_2,0)}(-1) - m\overline{V}_{D\cup D^*(\infty_2,\infty)}(-1).$$

By Theorem 4.1, we know that $V_{D\cup D^*(\infty_2,0)}(-1)$ and $\overline{V}_{D\cup D^*(\infty_2,\infty)}(-1) \equiv 1$ mod 8. Thus we have $m(V_{D\cup D^*(\infty_2,0)}(-1) - \overline{V}_{D\cup D^*(\infty_2,\infty)}(-1)) \equiv 0$ mod $8|m|$.

5. The minimal twisting number

In this section, we introduce the minimal twisting number for a knot with a symmetric union presentation.

Definition 5.1. We call the number $k-\mu$ of $D_K \cup D_K^*(\infty_\mu, m_{\mu+1}, \ldots, m_k)$ the *twisting number* of the symmetric union. The *minimal twisting number* of a link L with a symmetric union presentation is the smallest number of the twisting numbers of all symmetric union presentations for L. We denote it by $\mathrm{tw}(L)$.

By the definition, we have the following.

Proposition 5.2. *The minimal twisting number is an invariant of a symmetric union.*

Remark 5.3. Let L be a link with a symmetric union presentation. If $\mathrm{tw}(L) = 0$, then each component of L is a connected sum of a knot and its mirror image, could possibly be the unknot.

Example 5.4. For each knot K in $\{6_1, 8_8, 8_{20}, 9_{46}, 10_3, 10_{22}, 10_{35}, 10_{137}, 10_{140}, 10_{153}\}$, we have $\mathrm{tw}(K) = 1$. (See [6].)

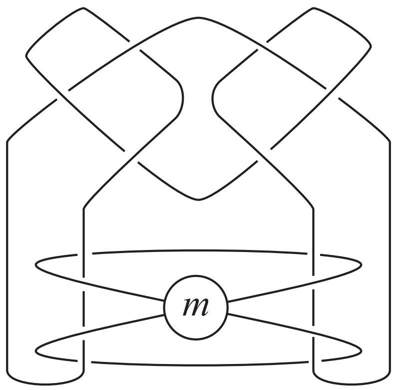
By definition, we can easily see the following.

Proposition 5.5. *Let $\overline{L}$ be a symmetric union with the two components K_1 and K_2. then K_1 and K_2 are symmetric uions and satisfies* $\mathrm{tw}(K_1) + \mathrm{tw}(K_2) \leq \mathrm{tw}(\overline{L})$.

Now we consider the following problem.

Problem. Does the equality of the inequality of Proposition 5.5 always hold?

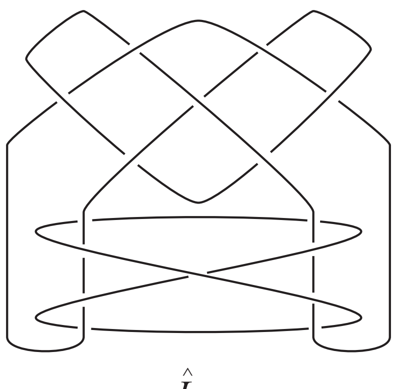
Example 5.6. Let L_m ($m \in \mathbb{Z}$, $m \neq 0$) be the symmetric union with two components K_1 and K_2 such that $\mathrm{tw}(K_1) = \mathrm{tw}(K_2) = 0$ as shown in Figure 3. We know that $\mathrm{tw}(L_m) = 1$. In fact, by Theorem 1.1, we have
$\overline{V}_{L_m}(t) = (-1)^m t^{-m}(3 - t^{-3} + t^{-2} - t^{-1} - t + t^2 - t^3) + (1 - (-1)^m t^{-m})(13 - t^{-5} + 3t^{-4} - 6t^{-3} + 9t^{-2} - 11t^{-1} - 11t + 9t^2 - 6t^3 + 3t^4 - t^5)$.
If $m > 0$, then the maximal degree is 5 and the minimal degree is $-m-5$. On the one hand, if $m < 0$, then the maximal degree is $5 - m$ and the minimal degree is -5. In particular, we know that L_m is not amphicheiral. Thus we have $\mathrm{tw}(L_m) = 1$.



L_m

FIGURE 3

Now we consider the following symmetric union $\hat{L}$ with two components K_1 and K_2 such that $\mathrm{tw}(K_1) = 1$ and $\mathrm{tw}(K_2) = 0$. We know that $\mathrm{tw}(\hat{L}) \geq 1$. We can show that $\hat{L}$ cannot have a



$\hat{L}$

FIGURE 4

presentation $D_K \cup D_K^*(\infty_2, m)$ if $|m| \neq 1$ by Theorem 1.2 and Corollary 1.5. However we do not know if $\hat{L}$ has a presentation $D_K \cup D_K^*(\infty_2, \pm 1)$ at this moment.

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