Conservative Automorphisms with zero Entropy

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1. Preliminary. Let $(\Omega, \mathfrak{B}, m)$ be a σ-finite measure space. A mapping T of Ω into itself is called an *endomorphism* if T is measurable and $m(AT^{-1}) = m(A)$ holds for all $A \in \mathfrak{B}$. Especially, if T is one-one mapping and T^{-1} is also endomorphism then T is called an *automorphism*. An endomorphism T is called *conservative*, if there exists no such set A (wandering set) of positive measure that A, AT^{-1}, $AT^{-2}, \cdots\cdots$ are mutually disjoint.

Let T be an endomorphism of the σ-finite measure space $(\Omega, \mathfrak{B}, m)$. For any $E \in \mathfrak{B}$ let $R_0(E) = \Omega \setminus E$; where $\Omega \setminus E$ denotes the complement of E with respect to Ω, and let $R_t(E) = E \cap E T^{-1} \bigcap_{i=1}^{t-1} (\Omega \setminus E) T^{-i}$ for $t \geqq 1$. $R_t(E)$ is the set of points of E returning to E at time t for the first time. We define $r_E(\omega)$ to be $=t$ on $R_t(E)$ and $=\infty$ on $(\Omega \setminus \bigcup_{t=1}^{\infty} R_t(E)) \cap E$ and $=o$ on $\Omega \setminus E$. Let

$$\omega T_E = \omega T^{r_E(\omega)} \text{ for } \omega \in E \cap (\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E T^{-i}) = E_{inf},$$
$$\omega T_E = \omega \qquad \text{for } \omega \notin E_{inf}.$$

For any $A \in \mathfrak{B}$ let $\mathfrak{B}_A = \{B \cap A : B \in \mathfrak{B}\}$ and m_A the measure defined on $\mathfrak{B}_A$ by $m_A(B) = m(A \cap B)$. If T is conservative, then it can be shown using the recurrence theorem[5] that T_E is an endomorphism of $(E, \mathfrak{B}_E, m)$. T_E is called the endomorphism induced by T on E.

Let $(E, \mathfrak{B}, m)$ be a finite (totally) measure space. Finite or countable partitions of E into disjoint measurable sets will be denoted by small greak letters. If $\xi_1, \xi_2, \cdots$, are countably many partitions then $\bigvee_{i=1}^{\infty} \xi_i$ denotes the σ-field generated by $\{\xi_i;\ i = 1, 2 \cdots\}$, If $\xi = \{A_i\}$ is a partition and T be an endomorphism of $(E, \mathfrak{B}, m)$ then $\xi\ T^{-n}$ denotes the partition $\{A_i\ T^{-n}\}$. It is known that the relative entropy $h(T, \xi)$ is defined as follows [3]:

$$h(T,\ \xi) = -\sum_{ij} m(A_i \cap B_j) \log \frac{m(A_i \cap B_j)}{m(B_j)} = \sum_{ij} m(A_i \cap B_j) \log\ m(A_i / B_j);$$

where $\{B_j\}$ denotes the partition $\bigvee_{i=1}^{\infty} \xi\ T^{-i}$ and the sum is taken over all atoms of the fields ξ, $\bigvee_{i=1}^{\infty} \xi\ T^{-i}$. The entropy $h(T, m)$ of the endomorphism T is defined by

$$h(T, m) = \sup h(T,\ \xi)\ ;$$

where the supremum is taken over all finite partitions or over all countable partitions with finite entropy. Now let T be a conservative endomorphism of a σ-finite measure space $(\Omega, \mathfrak{B}, m)$. The entropy $h(T)$ of T can be defined as follows [1]:

$$h(T) = \sup_{E \in \mathfrak{B},\ o < m(E) < \infty} h(T_E,\ m_E)$$

The purpose of this note is to show that the necessary condition for $h(T)=o$ by Krengel [1] is also sufficient in the case of automorphisms.

2. Continuity of relative entropy. Let $(E, \mathfrak{B}, m)$ be a finte (totally) measure space. We consider the classe of all conutable partitions (measurable) of all measurable subsets of E; where any finite partition is considered as a countably infinite partition adding to infinite null sets of E, and we shall denote this classe by $\mathfrak{p}$.

Let $\xi=\{E_1, E_2, \cdots\cdots\}$ and $\eta=\{F_1, E_2, \cdots\cdots\}$ be any countable partitions of the subsets A and B of E respectively. In what follows such subsetas A shall be called the base set of the partition. If $A=B$ (mod null sets; abbreviated *mod o*) and there exists a mapping from ξ onto η such that it is one to one and measure preserving, then we shall call that ξ and η are equivalent and denote it by $\xi\sim\eta$.

Suppose now that $\xi=\{E_1, E_2, \cdots\cdots\}$ and $\eta=\{F_1, F_2, \cdots\cdots\}$ are any elements of $\mathfrak{B}$, and define $inf \sum_{i=1}^{\infty} m(E_i+F_i^*)$; where $+$ denotes the Boolean sum in $\mathfrak{B}$ and the infinimum is taken over all partitions $\eta^*=\{F_1^*, F_2^*, \cdots\cdots\}$ which are equivalent to η. Similarly we can define $inf \sum_{i=1}^{\infty} m(F_j+E_j^*)$; where $\xi^*=\{E_1^*, E_2^*, \cdots\cdots\}$ denotes any partition which is equivalent to ξ. Then we have

$$inf \sum_i m(E_i+F_i^*)=inf \sum_j m(F_j+E_j^*).$$

For, from definition of equivalence of the partitions we have $F_i^*=F_{n_i}$ and n_i takes over all positive integers according as i runs over all ones. Hence if we put $n_i=j$, $i=m_j$ then $\sum_i m(E_i+F_i^*)=\sum_j m(F_j+E_j^*)$: where $E_j^*=E_{m_j}$, and the partition $\xi^*=\{E_1^*, E_2^*, \cdots\cdots\}$ is clearly equivalent to ξ.

Let $d(\xi, \eta)=inf \sum m(E_i+F_i^*)=inf \sum_j m(F_j+E_j^*)$, then $d(\xi, \eta)=d(\eta, \xi)$ and $d(\xi, \eta)=d(\xi^*, \eta^*)$ for $\xi^*\sim\xi$, $\eta^*\sim\eta$. Now, let $\bar{\mathfrak{p}}$ be the set of all equivalence classes of $\mathfrak{p}$, then we can define $d(\bar{\xi}, \bar{\eta})$ for $\bar{\xi}$, $\bar{\eta}\in\bar{\mathfrak{p}}$ by $d(\bar{\xi}, \bar{\eta})=d(\xi, \eta)$; where $\xi\in\bar{\xi}$, $\eta\in\bar{\eta}$ If we put $\bar{\xi}, \bar{\eta}\in\bar{\mathfrak{B}}$ such that $\bar{\xi}\neq\bar{\eta}$, then $d(\bar{\xi}, \bar{\eta})\neq o$. For, if $d(\bar{\xi}, \bar{\eta})=inf \sum_i m(E_i+F_i^*)=o$ then for any $E_i\in\xi\in\bar{\xi}$ there exists $F_i^*\in\eta\in\bar{\eta}$ such that $m(E_i+F_i^*)=o$ (otherwise, there exists a mutually disjoint infinite sequence $\{F_{nj}^*\}$ such that $m(E_i+F_{nj}^*)\longrightarrow o$. This implies that $m(F_{nj}^*)>\frac{1}{2}m(E_i)$, and $m(E)>m(\bigcup_j F_{nj}^*)=\infty$. Since $m(E)<\infty$, it is a contradiction). This implies that $\xi\sim\eta^*$, i.e., $\bar{\xi}=\bar{\eta}$. It contradicts to the hypothese. Now we put for any $\bar{\xi}\in\bar{\mathfrak{B}}$ and any $\varepsilon>o$

$$U(\bar{\xi};\ \varepsilon)=\{\bar{\eta}\in\bar{\mathfrak{p}};\ d(\bar{\xi}, \bar{\eta})<\varepsilon\},$$

then from the preceding results we can introduce a T_1-topology having the neighbourhood system $\{U(\bar{\xi};\ \varepsilon)\}$.

Now we consider the relative entropy $h(T_A, \xi)$ for acnservative endomorphism T on E and $\xi\in\mathfrak{p}$; where A is the base set of ξ and T_A denotes the endomorphism induced by T in A. It is easy to see from the definitions of relative entropy and equivalene of partitions that $h(T_A, \xi)=h(T_A, \xi^*)$ for $\xi\sim\xi^*$, i., we can defin $h(T_A, \bar{\xi})$ for $\bar{\xi}\in\bar{\mathfrak{p}}$ by $h(T_A, \bar{\xi}=h(T_A, \xi)$ for $\xi\in\bar{\xi}$.

Next, we shall show that $h(T_A, \bar{\xi})$ is a continuous function of $\bar{\xi}$ on $\bar{\mathfrak{B}}$ for fixed T, For any $\delta>o$ let $\bar{\eta}\in U(\bar{\xi};\delta)$ and $\xi=\{E_1,E_2,\cdots\cdots\}\in\bar{\xi}$ then there exists $\eta\in\bar{\eta}$; $\eta=\{F_1, F_2,\cdots\cdots\}$, such that $m(E_i+F_i)=\delta_i$, $\sum\delta_i<2\delta$. Let $\bigvee_{k=1}^{\infty}\xi\,T^{-k}=\{B_1,B_2,\cdots\cdots\}$, then for any B_j there exists subset $\{E_{j1},E_{j2},\cdots\cdots\}$ of ξ such that $B_j=E_{j1}\cap E_{j2}\;T_A^{-1}\cap E_{j3}\;T_A^{-2}\cap\cdots\cdots$ Suppose now the mapping

$$B_j\longrightarrow C_j=F_{j1}\cap F_{j2}\;T_B^{-1}\cap\;F_{j3}T_B^{-2}\cap\cdots\cdots;$$

where B is the dase set of η and T_B is the induced endomorphism of T in B. Then the mapping is clearly one to one from $\bigvee_{k=1}^{\infty}\xi\;T_A^{-k}$ onto $\bigvee_{k=1}^{\infty}\eta\;T_B^{-k}$. Moreover, since $(E_{jk}\cap F_{ik})T_A^{-k}=(E_{jk}\cap F_{jk})T_B^{-k}$ and consequently $m(E_{jk}\,T_A^{-k}+F_{jk}\,T_B^{-k}),\leqq m(E_{jk}+F_{jk})$ thus

$$m(B_j+C_j)\leqq\sum_{k=1}^{\infty}m(E_{jk}+F_{jk})=\sum_k\delta_{jk}\leqq\sum_j\delta<2\delta$$

Now we can show from the above results using the elemental computations that for any $\varepsilon>o$ there exists $\delta>o$ such that for any $\bar{\eta}\in U(\bar{\xi};\delta)$ $|\,h(T_A,\bar{\xi})-h(T_B,\bar{\eta})\,|<\varepsilon$; where A and B denote the base sets of $\xi\in\bar{\xi}$ and $\eta\in\bar{\eta}$ respectively. More precisely, there exists $\eta=\{F_1,F_2,\cdots\cdots\}\in\bar{\eta}$ such that

$$|\sum_{ij}m(E_i\cap B_j)log\;m(E_i\,|\,B_j)-\sum_{ij}m(F_i\cap C_j)log\;m(F_i\,|\,C_j)\,|<\varepsilon;$$

where $\xi=\{E_1,\;E_2,\cdots\cdots\}$, $B_j=E_{j1}\cap E_{j2}T_A^{-1}\cap\cdots\cdots\in\bigvee_{k=1}^{\infty}\xi T_A^{-k}$ and $C_j=F_{j1}\cap F_{j2}T_B^{-1}\cap\cdots\cdots\cdots\in\bigvee_{k=1}^{\infty}\eta T_B^{-k}$. Thus we obtain the continuity of the relative entropy $h(T_A,\bar{\xi}.)$

3. Conservative automorphisms with zero entropy. Let $(\Omega,\mathfrak{B},m)$ be a σ-finite measure space and T be a conservative endomorphism on Ω. It is known that $h(T)=o$ implies the following condition [1] :

(*) For any countable partition ξ of the form $\xi=\{A_1,A_2,\cdots\cdots,\Omega/A\}$ with $m(A)<\infty$ and $-\sum_i\overline{m}_A(A_i)log\;\overline{m}_A(A_i)<\infty$; where $A=\bigcup_{i=1}^{\infty}A_i$ and $\overline{m}_A$ denotes the normalized measure of m_A, the following relation holds

$$\xi\subseteqq\bigvee_{k=1}^{\infty}\xi\,T^{-k}.$$

In this section we shall show that the above necessary condition (*) of $h(T)=o$ is also sufficient in the case of automorphisms. of course, it is known that this condition (*) is sufficient in the case $m(\Omega)=1$ even when T is an endomorphism [4, 1].

First we show the following lemma which is essentially the same as the proposition given by Krengel [1, 2] with respect to quasi-finite automorphisms.

LEMMA. *Let T be a conservative automorphism on a σ-finite measure space $(\Omega,\mathfrak{B},m)$ satisfying the above condition (*). For any $E\in\mathfrak{B}$; $m(E)<\infty$, if the partition $\rho_E=\{R_1(E),\;R_2(E),\cdots\cdots\}$ has finite entropy then $h(T_E,\;m_E)=o$.*

PROOF. Let ξ_E be any countable partition of E with finite entropy. We can suppose no loss of generality that ξ_E finer than $\rho_E T_E$, i.e., $\rho_E\subseteqq\xi_E T_E^{-1}$. Now, ξ denotes the partition of Ω constituted with $\Omega\;E$ and the sets of ξ_E. Let A be any atom of ξ. If $A\subseteqq E$ and for simplicity R_i denotes $R_i(E)$ then

$$AT^{-n}\cap E=\bigcup_{j=1}^{\infty}\bigcup_{i1+\cdots+ij=n}R_{i1}\cap R_{i2}\;T_E^{-1}\cap\cdots\cap R_{ij}\;T_E^{-(j-1)}\cap AT_E^{-j}\in\bigvee_{k=1}^{\infty}\xi_E T_E^{-k}.$$

If $A=\Omega\diagdown E$ then

$$AT^{-n}\cap E=\bigcup R_{i_1}\cap R_{i_2}T_E^{-1}\cap\cdots\cap R_{i_n}T_E^{-(n-1)}\in\bigvee_{l=1}^{\infty}\xi_E T_E^{-k};$$

where the union is taken over all pairs $(i_1, i_2, \cdots\cdots, i_n)$ with the properties $i_j \geqq 1$ $(j=1, 2, \cdots\cdots, n)$ and $i_1+\cdots+i_k \neq n$ for all $k\leqq n$. These results show that $E\cap\bigvee_{k=1}^{\infty}\xi T^{-k}\subset\bigvee_{k=1}^{\infty}\xi_E T_E^{-k}$. Since $\xi\subset\bigvee_{k=1}^{\infty}\xi T^{-k}$ (condition (*)) and $\xi_E=E\cap\xi$, hence we have $\xi_E\subset\bigvee_{k=1}^{\infty}\xi_E T_E^{-k}$. It implies that $h(T_E, \xi_E)=o$ for any ξ_E, and consequently $h(T_E, m_E)=sup\ h(T_E, \xi_E)=o$.

We shall now show the sufficiency of the condition (*) for $h(T)=o$.

THEOREM. *Let* $(\Omega, \mathfrak{B}, m)$ *be a* σ*-finite measure space and* T *be a conservative automorphism on* Ω. *If* T *satisfies the condition* (*), *then for any* $E\in\mathfrak{B}$; $m(E)<\infty$, $h(T_E, m_E)=o$, *i.e.*, $h(T)=o$.

PROOF. It is clear from the preceding lemma that $h(T_E, m_E)=o$ for the case E having the partition ρ_E with finite e ntropy. Hence it sufficies to consider the case that ρ_E has not finite entropy. Let $\rho_E=\{R_1(E), R_2(E), \cdots\}$ and $E_k^*=\bigcup_{=1}^{k}R_i(E)$. Since $m(E)<\infty$, for any $\varepsilon>o$ there exists an integer k_0 such that for any integer $k\geqq k_0$ $m(E\setminus E_k^*)<\varepsilon$. Suppose now that the partition $\rho_{E_k^*}=\{R_1(E_k^*), R_2E_k^*), \cdots\}$ of E_k^*. Then it is clear that $R_i(E_k^*)\subset R_i(E)$. If we put $S_i=R_i(E)\setminus R_i(E_k^*)$, then it is an immediate concequence from the definition of S_i that $S_iT_E^i\subset E\setminus E_k^*$and $S_iT_E^l\cap E=\phi$ (mod o); where ϕ denotes the null set. Moreover, if $i\neq j$ (let $i<j$) and $S_iT_E^i\cap S_j\ T_E^j$ contains a set A having non-zero measure, then $AT^{-i}\subset E$ (since $S_i\subset R_i(E)\subset E$) and $AT^{-i}\subset S_j T_E^{j-i}$, i.e., $S_jT_E^{j-i}\cap E\neq\phi$. It is a contradiction. Hence $S_iT_E^i\cap T_jT_E^j=\phi$ $(mod\ o)$ for $i\neq j$. Consequently, we have $m(\bigcup^{k}_{=1}S_i)=\sum m(S_i)=\sum m(S_iT_E^i)=m(\bigcup^{k}_{=1}S_iT_E^i)<m(E\setminus E_k^*)<\varepsilon$, i.e., $m(\bigcup_{i=1}^{k}S_i)<\varepsilon$. From this result it is easy to see by the elemental computations that for any $\varepsilon>o$ there exists an integer k_0 Such that

$$\left|\sum_{i=1}^{\infty}m(R_i(E_k^*))log\ m(R_i(E_k^*))-\sum_{i=1}^{k}m(R_i(E))\ log\ m(R_i(E))\right|<\varepsilon \text{ for } k\geqq k_0.$$

It implies that the partition $\rho_{E_k^*}$ has finite entropy. Now let ξ_E be any countable partition of E with finite entropy. Then the partition $\xi_{E_k^*}$ of E_k^* induced by ξ_E in E_k^* has finite eintropy. From the preceding lemma we have $h(T_{E_k^*}, \xi_{E_k^*})=o$. While, let $\xi_E=\{F_1, F_2, \cdots\}$ and $\xi_{E_k^*}=\{F_1^*, F_2^*, \cdots\}$; where $F_i^*=F_i\cap E_k^*$, then $\bigcup_{i=1}^{\infty}(F_i+F_i^*)=\bigcup_{i=1}^{\infty}(F_i\setminus F_i^*)=E\setminus E_k^*$. Therefore, $\sum_{=1}^{\infty}m(F_i+F_i^*)=m(E\setminus E_k^*)<\varepsilon$. It implies that $\bar{\xi}_{E_k^*}$ converges to $\bar{\xi}_E$ for $k\to\infty$ in $\bar{\mathfrak{p}}$. From the continuity of relative entropy we have $h(T_{E_k^*}, \bar{\xi}_{E_k^*})$ converges to $h(T_E, \bar{\xi}_E)$. Since $h(T_{E_k^*}, \bar{\xi}_{E_k^*})=o$ for sufficiently large integer k, hence $h(T_E, \xi_E)=o$. It implies that $h(T_E, m_E)=o$. Consequently, we have $h(T)=sup\ h(T_E, m_E)=o$. This completes the proof.

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